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LETTER TO THE EDITOR

A systematic way of converting infinite series into infinite products

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Abstract. It is shown how to display q-series in the forms $\Sigma(\pm 1)^n q^{(kn+l)^2}$ and $\Sigma(-1)^{n(n\pm 1)/2} q^{(kn+l)^2}$ as infinite products. Systematic ways of expanding q-series and q-products into other series and products are also given. A method for determining the independence, or otherwise, of different q-series is discussed.

In many models in lattice statistics exact solutions of certain properties appear as infinite products of the sort that occur in the expansion of elliptic functions, e.g. Baxter *et al* (1975), Baxter (1980) and Pearce (1985). Although infinite products are easily dealt with using computers, it is often desirable to find a series expansion of the parameter involved, in which case expressions in terms of infinite series may be preferred. Here, it will be shown how to convert infinite series that occur in the theory of elliptic functions into infinite products, thus indicating how the reverse procedure might be accomplished. This will be demonstrated in several cases. Relations between infinite series and infinite products have an honourable history starting with Euler and Gauss and first studied systematically by Jacobi. Some of these results will be generalised and new ones obtained. Inevitably a new notation will be introduced, but relations between the new and conventional notations will be given. A method will be described by which the independence of various series can be determined.

Our series notation is as follows. Let

$$\theta(k, l:q) = \sum q^{(kn+l)^2} \qquad \bar{\theta}(k, l) = \sum (-1)^n q^{(kn+l)^2}$$
$$\phi(k, l) = \sum (-1)^{n(n-1)/2} q^{(kn+l)^2} \qquad \bar{\phi}(k, l) = \sum (-1)^{n(n+1)/2} q^{(kn+l)^2} \qquad (1)$$

where Σ implies summation over all n from $-\infty$ to $+\infty$. The following properties of these functions are easily deduced. If k and l are not relatively prime but have a common factor r, then for all the functions, f, defined by (1)

$$f(rk, rl) = f(k, l:q^{r^2}). \qquad (2)$$

In conventional notation three θ-series often occur. These are

$$\theta_3 = \sum_{-\infty}^{\infty} q^{n^2} \qquad \theta_4 = \sum_{-\infty}^{\infty} (-1) q^{n^2} \qquad \theta_2 = \sum_{-\infty}^{\infty} q^{(n-1/2)^2}.$$

In the new notation

$$\theta_3 = \theta(1, 0) \qquad \theta_4 = \bar{\theta}(1, 0) \qquad \theta_2 = \theta(1, \tfrac{1}{2}).$$

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The expansion properties are:

$$\phi(k, l) = \bar{\theta}(2k, l) + \bar{\theta}(2k, k+l) \qquad \bar{\phi}(k, l) = \bar{\theta}(2k, l) - \bar{\theta}(2k, k+l) \tag{3}$$

$$\theta(k, l) = \theta(2k, l) + \theta(2k, k+l) = \theta(3k, l) + \theta(3k, k+l) + \theta(3k, 2k+l)$$
$$= \theta(nk, l) + \theta(nk, k+l) + \ldots + \theta(nk, (n-1)k+l) \tag{4}$$

$$\bar{\theta}(k, l) = \theta(2k, l) - \theta(2k, k+l) = \bar{\theta}(3k, l) - \bar{\theta}(3k, k+l) + \bar{\theta}(3k, 2k+l) \tag{5}$$

etc.
The reflection properties are:

$$\theta(k, l) = \theta(k, k-l) \qquad \bar{\theta}(k, l) = -\bar{\theta}(k, k-l) \tag{6}$$

$$\phi(k, l) = -\phi(k, k-l) \qquad \bar{\phi}(k, l) = \bar{\phi}(k, k-l). \tag{7}$$

Our products notation is as follows. Let

$$Q(k, l) = \prod(1 - q^{kn-l}) \qquad \bar{Q}(k, l) = \prod(1 + q^{kn-l}) \tag{8}$$

$$Q(rk, rl) = Q(k, l : q^r) \qquad \bar{Q}(rk, rl) = \bar{Q}(k, l : q^r) \tag{9}$$

where Π implies product over all n from 1 to ∞.

In conventional notation the following products often occur.

$$Q_0 = \prod(1 - q^{2n}) \qquad Q_1 = \prod(1 + q^{2n})$$

$$Q_2 = \prod(1 + q^{2n-1}) \qquad Q_3 = \prod(1 - q^{2n-1}).$$

In this new notation

$$Q_0 = Q(2, 0) \qquad Q_1 = \bar{Q}(2, 0) \qquad Q_2 = \bar{Q}(2, 1) \qquad Q_3(2, 1).$$

A useful identity is that $Q_1 Q_2 Q_3 = 1$.
The expansion properties are:

$$Q(k, l) = Q(2k, l)Q(2k, k+l) = Q(3k, l)Q(3k, k+l)Q(3k, 2k+l) \ldots \tag{10}$$

and similarly with $\bar{Q}$.
The combining property is

$$Q(k, l)\bar{Q}(k, l) = Q(2k, 2l) = Q(k, l : q^2). \tag{11}$$

The reduction properties are:

$$Q(k, k) = 0 \qquad \bar{Q}(k, k) = 2Q(k, 0) \tag{12}$$

$$Q(k, l) = (1 - q^{k-l})Q(k, l-k) \qquad \bar{Q}(k, l) = (1 + q^{k-l})\bar{Q}(k, l-k). \tag{13}$$

For the conversion of series into products, we start with the fundamental Jacobi identity which may be written

$$\sum x^n q^{n^2} = \prod(1 - q^{2n})(1 + xq^{2n-1})(1 + x^{-1}q^{2n-1}). \tag{14}$$

Replacing q by q^{k^2}, x by q^{2kl} and multiplying both sides by q^{l^2}, we immediately obtain

$$\theta(k, l) = q^{l^2} Q(2k, 0)\bar{Q}(2k, k+2l)\bar{Q}(2k, k-2l)|q^k \tag{15}$$

where $|q^k$ implies that all the previous functions have argument q^k in place of q. Using similar but more complicated substitutions we deduce that

$$\bar{\theta}(k, l) = q^{l^2} Q(2k, 0)Q(2k, k+2l)Q(2k, k-2l)|q^k \tag{16}$$

$$\phi(k, l) = q^{l^2} Q(4k, 0) \bar{Q}(4k, 2k) Q(4k, k-2l) \times \bar{Q}(4k, 3k-2l) \bar{Q}(4k, k+2l) Q(4k, 3k+2l)|q^k \quad (17)$$

$$\bar{\phi}(k, l) = q^{l^2} Q(4k, 0) \bar{Q}(4k, 2k) \bar{Q}(4k, k-2l) \times Q(4k, 3k-2l) Q(4k, k+2l) \bar{Q}(4k, 3k+2l)|q^k. \quad (18)$$

Is the inverse problem solvable? That is, can $Q(k, l)$ be expressed in terms of various q-series? It seems unlikely, though the following result is simply obtained. From (16) we may write

$$\begin{aligned} \bar{\theta}(6, 1) &= qQ(12, 0)Q(12, 8)Q(12, 4)|q^6 \\ &= qQ(3, 0)Q(3, 2)Q(3, 1)|q^{24} \qquad \text{(using (9))} \\ &= qQ(1, 0: q^{24}) \qquad \text{(using (10))}. \end{aligned}$$

Thus in expanded notation this is

$$\sum (-1)^n q^{(6n+1)^2} = q\prod(1-q^{24n}) \quad (19)$$

which is Euler's result. This so far seems the only case in which a single infinite product is expressible as a single infinite series. However, many combinations of infinite products appearing in lattice statistics solutions may be expressed as ratios of infinite series. For example, consider (Baxter 1980)

$$q^{1/5} g(q) = q^{1/5} \prod \frac{(1-q^{5n-4})(1-q^{5n-1})}{(1-q^{5n-3})(1-q^{5n-2})}.$$

By using (16) and expanding $\bar{\theta}(10, 1)$ and $\bar{\theta}(10, 3)$ it is easily shown that

$$q^{1/5} g(q) = \frac{\bar{\theta}(10, 3)}{\bar{\theta}(10, 1)} = \frac{\Sigma(-1)^n q^{(10n+3)^2/40}}{\Sigma(-1)^n q^{(10n+1)^2/40}}. \quad (20)$$

Actually this result is implicit in Hardy and Wright (1960, theorems 355 and 356) but without the $q^{1/5}$ they do not have the pleasing form exhibited above. Similarly an infinite product result found in Baxter *et al* (1975) is expressible as a ratio of two series. This is

$$q^{1/2} \prod \frac{(1-q^{8n-1})(1-q^{8n-7})}{(1-q^{8n-3})(1-q^{8n-5})} = \frac{\Sigma(-1)^n q^{(2n+3/4)^2}}{\Sigma(-1)^n q^{(2n+1/4)^2}}. \quad (21)$$

A further result relating a product given by Baxter (1980) to a ratio of series is

$$q^{3/5} \prod \frac{(1-q^{5n})^2}{(1-q^{5n-2})(1-q^{5n-3})} = \frac{\Sigma(4n+1) q^{5(4n+1)^2/8}}{\Sigma(-1)^n q^{(10+1)^2/40}}. \quad (22)$$

This result, however, belongs to the theory of derivatives of $\theta(k, l)$; that is

$$\theta'(k, l) = \sum (kn+l) q^{(kn+l)^2}$$

etc, which has not yet been fully developed. The one well known result is Jacobi's famous theorem which in conventional notation is (Hardy and Wright 1960)

$$\prod (1-q^n)^3 = \sum_{m=0}^{\infty} (-1)^m (2m+1) q^{m(m+1)/2}.$$

In the notation developed above this appeared in the more symmetric form

$$qQ^3(8, 0) = \theta'(4, 1)$$

i.e.

$$q\prod(1-q^{8n})^3=\sum(4n+1)q^{(4n+1)^2}=\theta'(4,1) \tag{23}$$

and thus (22) was found. Recently, however, several new relations similar to Jacobi's but *independent* of his have appeared or will appear in print: Glasser and Zucker (1980), Borwein and Borwein (1986), Borwein and Zucker (1986), and we display them here in forms which are by no means unique:

$$q\prod(1-q^{16n})^3(1+q^{16n-8})^3=\sum(-1)^n(4n+1)q^{(4n+1)^2}=\bar{\theta}'(4,1) \tag{24}$$

$$q\prod(1-q^{6n})^3(1-q^{6n-3})^2(1+q^{6n})^2=\sum(3n+1)q^{(3n+1)^2}=\theta'(3,1) \tag{25}$$

$$q\prod(1-q^{6n})^3(1+q^{3n})^2=\sum(-1)^n(3n+1)q^{(3n+1)^2}=\bar{\theta}'(3,1) \tag{26}$$

$$q\prod(1-q^{48n})^3(1-q^{48n-24})^5=\sum(6n+1)q^{(6n+1)^2}=\theta'(6,1) \tag{27}$$

$$q\prod(1-q^{48n})^3(1+q^{48n-24})^5=\sum(-1)^{n(n-1)/2}(6n+1)q^{(6n+1)^2}=\phi'(6,1). \tag{28}$$

(25), (26) and (27) are not independent of one another. How this may be shown is indicated below.

The many additive and multiplicative relations which exist among the conventional functions θ_2, θ_3 and θ_4 may be easily demonstrated using (3)-(13). For example, using (4) one has

$$\theta(1,0)=\theta(2,0)+\theta(2,1)=\theta(1,0\text{: }q^4)+\theta(1,\tfrac{1}{2};\,q^4)$$

or, conventionally,

$$\theta_3=\theta_3(q^4)+\theta_2(q^4).$$

Clearly in this simple case it is seen that the θ-functions involved are not independent of one another, and that $\theta(2,1)$ is expressible in terms of $\theta(1,0)$. The question arises of how to ascertain in general when a given $\theta(k,l)$ is independent or not. Now this might be accomplished by manipulating (3)-(13). After some work it would be shown that $\theta(3,1)$, $\bar{\theta}(3,1)$, $\theta(3,2)$, $\bar{\theta}(3,2)$ and $\theta(4,1)$ are all expressible in terms of $\theta(1,0)$. However, $\bar{\theta}(4,1)$ does not succumb in a similar way, but it is not obvious why it should be regarded as a new independent series. The following analysis, however, makes it clear whenever a given series is linearly independent of another. Consider the Mellin transform of $f(t)$ defined by

$$M_s[f(t)]=\frac{1}{\Gamma s}\int_0^\infty t^{s-1}f(t)\,\mathrm{d}t. \tag{29}$$

If in the θ-series considered e^{-t} is written for q, then the Mellin transform of the θ-series become a Dirichlet L-series (Zucker and Robertson 1976). For example

$$M_s[\theta(1,0)-1]=2(1^{-2s}+2^{-2s}+3^{-s}\ldots)=2L_1(2s). \tag{30}$$

The Mellin transforms of $\bar{\theta}(1,0)\ldots\theta(4,1)$ are all expressible in terms of $L_1(2s)$ but

$$M_s[\bar{\theta}(4,1)]=1^{-2s}-3^{-2s}-5^{-2s}+7^{-2s}\ldots=L_8(2s) \tag{31}$$

which is an L-series algebraically independent of L_1. It thus becomes immediately clear that $\bar{\theta}(4, 1)$ is not expressible in terms of $\theta(1, 0)$. It is for this reason that one can state that (25), (26) and (27) are not independent series whereas the others are since if we take Mellin transforms of (23)-(28) we obtain

$$M_s\theta'(4, 1) = L_{-4}(2s-1) \tag{32}$$

$$M_s\bar{\theta}'(4, 1) = L_{-8}(2s-1) \tag{33}$$

$$M_s\theta'(3, 1) = L_{-3}(2s-1) \tag{34}$$

$$M_s\bar{\theta}'(3, 1) = (1+2^{2-2s})L_{-3}(2s-1) \tag{35}$$

$$M_s\theta'(6, 1) = (1+2^{1-2s})L_{-3}(2s-1) \tag{36}$$

$$M_s\phi'(6, 1) = L_{-24}(2s-1). \tag{37}$$

These results also enable one to evaluate many more three-dimensional lattice sums by the methods discussed by Glasser and Zucker (1980). For example

$$\sum\sum\sum \frac{(-1)^p}{[8m^2+8n^2+(4p+1)^2]^s} = L_{-8}(2s-1) \tag{38}$$

$$\sum\sum\sum \frac{(-1)^{m+n+p}}{[24m^2+24n^2+(6p+1)^2]^s} = (1+2^{1-2s})L_{-3}(2s-1) \tag{39}$$

$$\sum\sum\sum \frac{(-1)^{p(p-1)/2}}{[24m^2+24n^2+(6p+1)^2]^s} = L_{-24}(2s-1). \tag{40}$$

When s is an integer all the above results can be expressed in terms of algebraic numbers and powers of π. For example, for $s=1$, equation (39) gives $\sqrt{3}\,\pi/6$. A fuller report of these results will be given elsewhere.

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